

On Dynamic Programming Decompositions of Static Risk Measures in Markov Decision Processes

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Summary

Primal methods

Augment state space to keep track of target value or time. [Wu, 1999], [Lin, 2003], [Bauerle, 2011], [Hau, 2023]

Robust methods

Define **robust MDP** for risk measure with augmented state space to keep track of risk levels. These popular dynamic program (DP) methods for solving risk averse MDP [Chow, 2015], [Jin, 2019], [Li, 2022], [Ni, 2022] are thought to be optimal with sufficiently discretized risk levels for policy optimization.

Contribution

Policy Evaluation

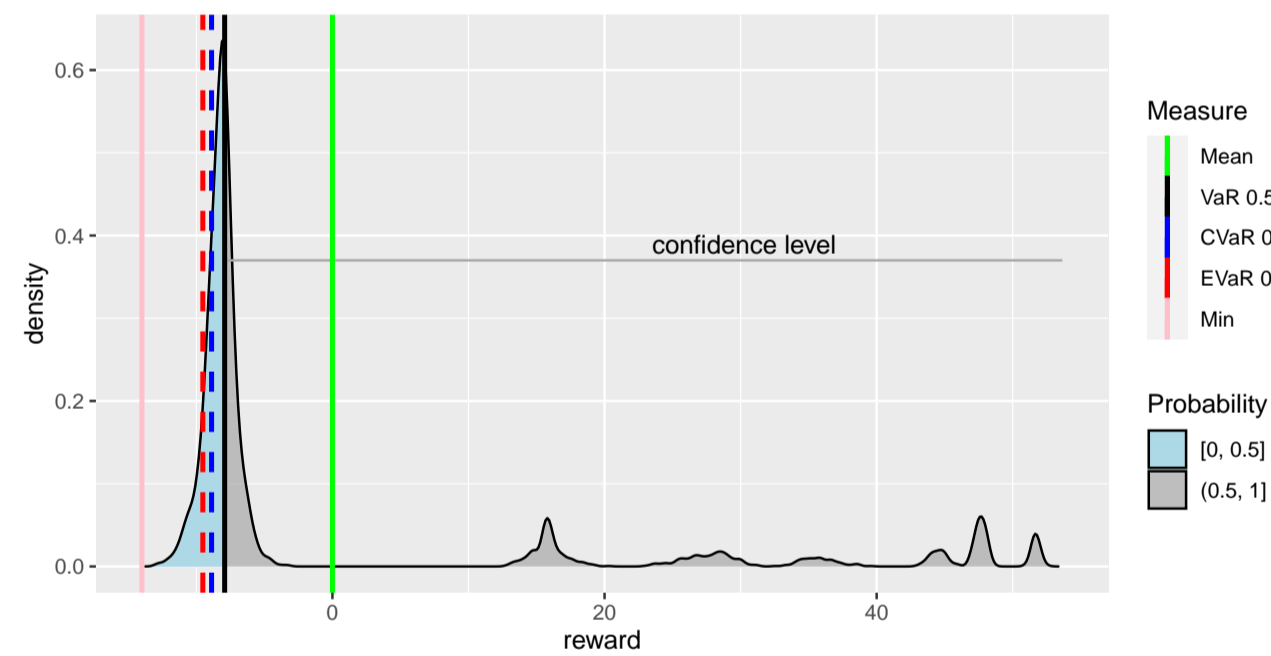
- Prove a new correct VaR and EVaR decomposition.

Policy Optimization

- Prove robust methods are suboptimal for CVaR [Chow 2015] and EVaR [Ni 2022] MDP.
- Propose optimal DP for robust methods Value-at-Risk (VaR) MDP.

Risk Averse MDPs

Risk Measures



$$\text{VaR}_\alpha[\tilde{x}] = \sup_{z \in \mathbb{R}} \{z \in \mathbb{R} \mid \mathbb{P}[\tilde{x} < z] \leq \alpha\} = \inf_{z \in \mathbb{R}} \{z \in \mathbb{R} \mid \mathbb{P}[\tilde{x} \leq z] > \alpha\}$$

$$\text{CVaR}_\alpha[\tilde{x}] = \sup_{z \in \mathbb{R}} (z - \alpha^{-1} \mathbb{E}[z - \tilde{x}]_+) = \inf_{\xi \in \Delta_m} \{\xi^T x \mid \alpha \cdot \xi \leq q\}$$

$$\text{EVaR}_\alpha[\tilde{x}] = \sup_{\beta > 0} -\frac{1}{\beta} \log(\alpha^{-1} \mathbb{E}[\exp(-\beta \cdot \tilde{x})]) = \inf_{\xi \in \Delta_m} \{\xi^T x \mid \text{KL}(\xi \parallel q) \leq -\log \alpha\}$$

Risk Averse Objectives

- Maximizes the risk measure $\psi[\cdot]$ of the total reward in a Markov decision process (MDP)

$$\max_{\pi \in \Pi} \psi \left[\sum_{t=0}^T r^\pi(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}) \right]$$

Assume: Rewards $r(s, a, s') \in \mathbb{R}$, transition probabilities $P(s, a) \in \Delta^S$ and finite horizon.

- If $\psi[r(\tilde{s}, \tilde{a}, \tilde{s}')] = \psi[r(s, \tilde{a}, \tilde{s}' \mid \tilde{s} = s)]$ and $\max_{\pi \in \Pi} \psi[r^\pi(\tilde{s}, \tilde{a}, \tilde{s}')] = \max_{\pi \in \Pi} \psi[r^\pi(s, \tilde{a}, \tilde{s}' \mid \tilde{s} = s)]$ then exist DP for evaluation and policy optimization respectively.

Challenges: $\psi \in \text{VaR, CVaR and EVaR}$, do not satisfy *tower property*: $\psi[X] = \psi[\psi[X \mid Y]]$.

$$\max_{\pi \in \Pi} \psi \left[\sum_{t=\tau}^T r^\pi(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}) \right] \neq \max_{a \in A} \left[r(s, a, s') + \max_{\pi \in \Pi} \psi \left[\sum_{t=\tau+1}^T r^\pi(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}) \mid \tilde{s}_{\tau+1} = s' \right] \right]$$

Extended Conditional Coherent Risk Measure

Proposition 3.1 : Lemma 22 in [Pflug, 2016] Suppose that $\pi \in \Pi$ and $\tilde{s} \sim \hat{p}$, $\tilde{a} \sim \pi(\tilde{s})$, $\tilde{s}' \sim p_{s,a}$. Then,

$$\text{CVaR}_\alpha[r(\tilde{s}, \tilde{a}, \tilde{s}')] = \min_{\zeta \in \mathcal{Z}_C} \sum_{s \in \mathcal{S}} \zeta_s \text{CVaR}_{\alpha \zeta_s \hat{p}_s^{-1}}[r(s, \tilde{a}, \tilde{s}') \mid \tilde{s} = s],$$

where the state s on the right-hand side is not random and $\mathcal{Z}_C = \{\zeta \in \Delta_S \mid \alpha \cdot \zeta \leq \hat{p}\}$.

CVaR Decomposition Fails in Optimization

Theorem 3.2 : There exists an MDP and a risk level $\alpha \in [0, 1]$ such that

$$\begin{aligned} \max_{\pi \in \Pi} \text{CVaR}_\alpha[r^\pi(\tilde{s}, \tilde{a}, \tilde{s}')] &= \max_{\pi \in \Pi} \min_{\zeta \in \mathcal{Z}_C} \sum_{s \in \mathcal{S}} \zeta_s \text{CVaR}_{\alpha \zeta_s \hat{p}_s^{-1}}[r^\pi(s, \tilde{a}, \tilde{s}') \mid \tilde{s} = s] \\ &< \min_{\zeta \in \mathcal{Z}_C} \max_{\pi \in \Pi} \sum_{s \in \mathcal{S}} \zeta_s \text{CVaR}_{\alpha \zeta_s \hat{p}_s^{-1}}[r^\pi(s, \tilde{a}, \tilde{s}') \mid \tilde{s} = s] \\ &= \min_{\zeta \in \mathcal{Z}_C} \sum_{s \in \mathcal{S}} \zeta_s \max_{\pi \in \Pi} \text{CVaR}_{\alpha \zeta_s \hat{p}_s^{-1}}[r^\pi(s, \tilde{a}, \tilde{s}') \mid \tilde{s} = s] \end{aligned}$$

Let $\theta_\pi(\zeta) = \sum_{s \in \mathcal{S}} \zeta_s \text{CVaR}_{\alpha \zeta_s \hat{p}_s^{-1}}[r^\pi(s, \tilde{a}, \tilde{s}') \mid \tilde{s} = s]$ and $\alpha = 0.5$.

Figure right below plot the function $\min_{\zeta \in \mathcal{Z}_C} \max_{\pi \in \Pi} \theta_\pi(\zeta)$ and $\max_{\pi \in \Pi} \min_{\zeta \in \mathcal{Z}_C} \theta_\pi(\zeta)$ for the 2 states simple example in left below to illustrate the inaccurate approximation.

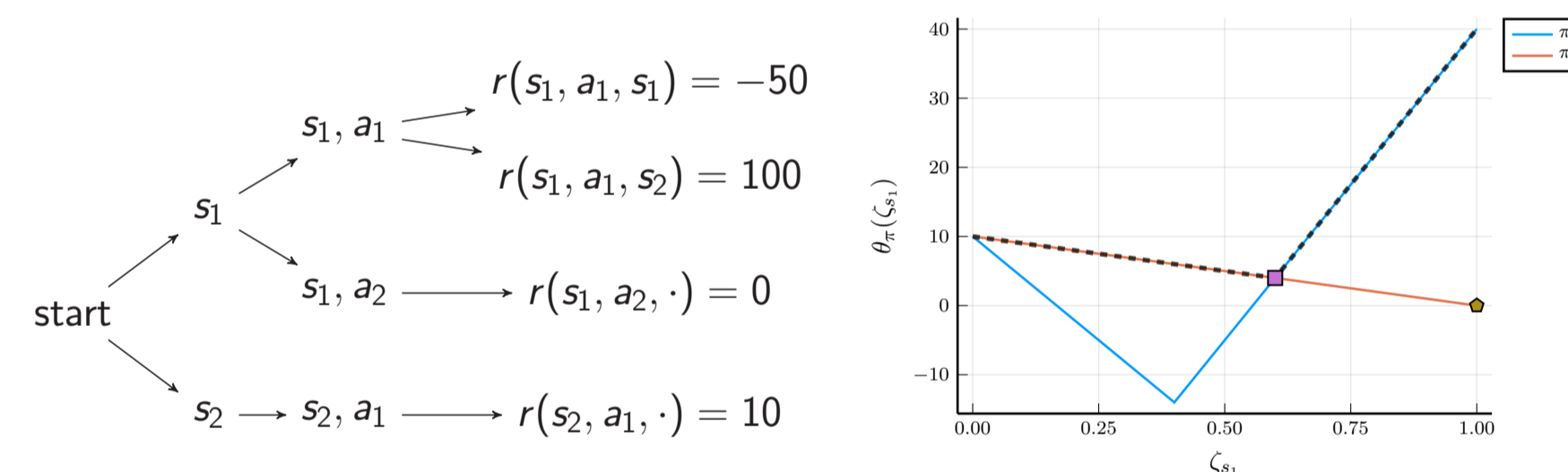
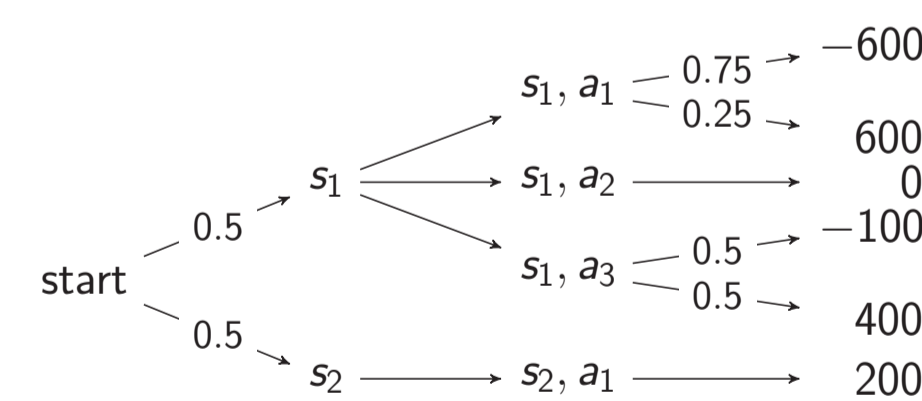
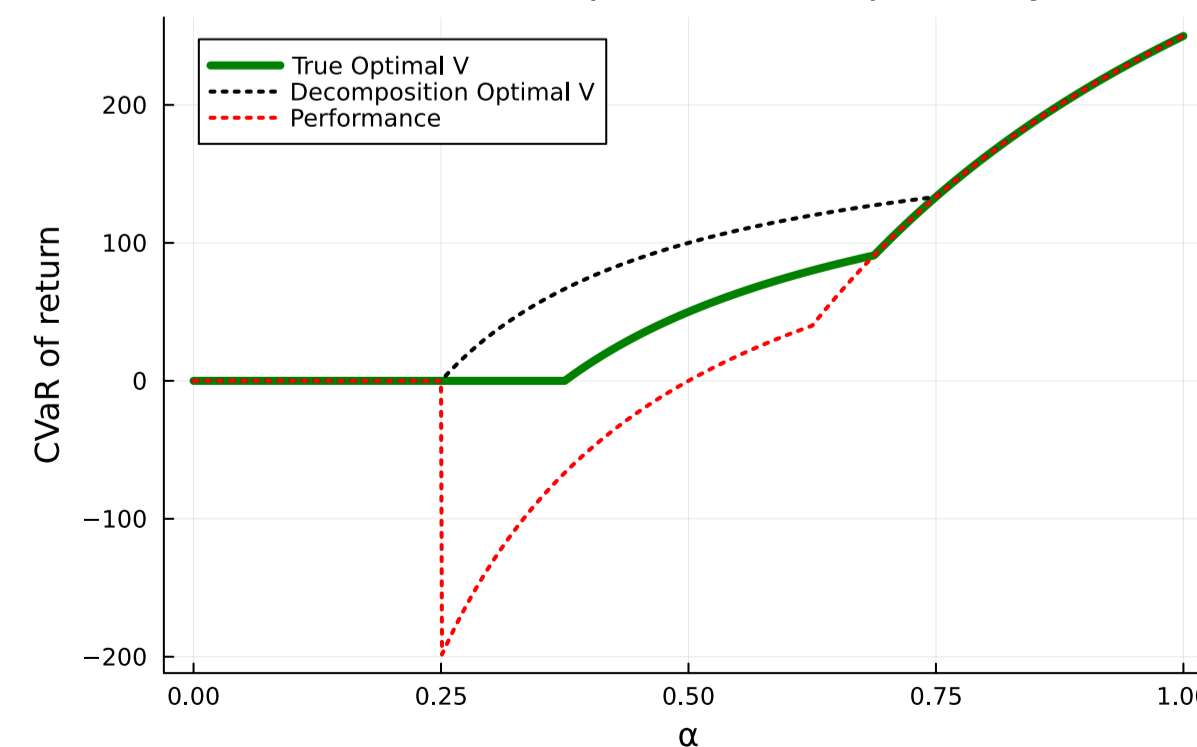


Figure below demonstrate an example and illustrate the sub-optimality of a policy for CVaR decomposition.



CVaR decomposition suboptimality



EVaR Decomposition Fails in Optimization

Theorem 4.2: EVaR evaluation decomposition Given $\alpha \in (0, 1]$, we have that

$$\text{EVaR}_\alpha[r(\tilde{s}, \tilde{a}, \tilde{s}')] = \inf_{\zeta \in \mathcal{Z}_E(\alpha)} \sum_{s \in \mathcal{S}} \zeta_s \text{EVaR}_{\zeta_s} [r(s, \tilde{a}, \tilde{s}') \mid \tilde{s} = s],$$

where $\mathcal{Z}_E(\alpha) = \{\zeta \in \Delta_S \mid \xi \ll \hat{p}, \sum_{s \in \mathcal{S}} \zeta_s (\log(\zeta_s / \hat{p}_s) - \log(\zeta_s)) \leq -\log \alpha\}$.

Note : EVaR Decomposition also fails in optimization with the same reason in Theorem 3.2.

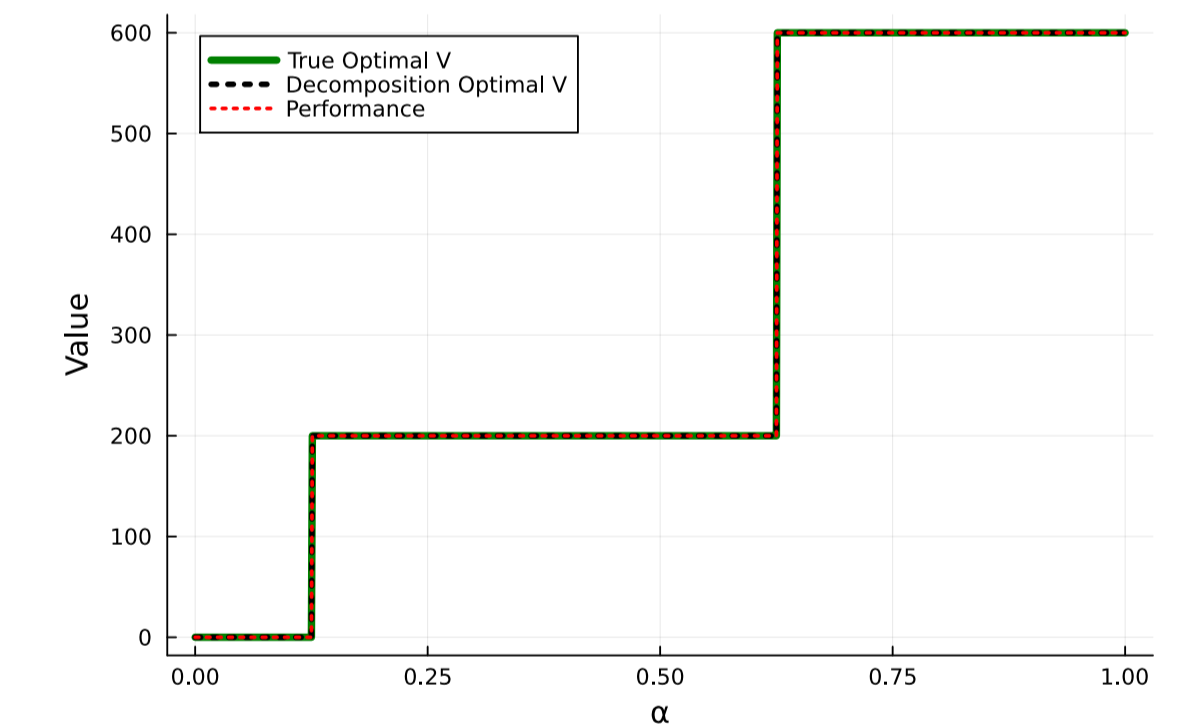
Value at Risk (VaR) MDP

Theorem 5.1: VaR policy evaluation Given any finite MDP for risk level $\alpha \in [0, 1]$ we have

$$\text{VaR}_\alpha[r(\tilde{s}, \tilde{a}, \tilde{s}')] = \sup_{\zeta \in \Delta_N} \left\{ \min_s \text{VaR}_{\alpha \zeta_s \hat{p}_s^{-1}} [r(s, \tilde{a}, \tilde{s}') \mid \tilde{s} = s], \alpha \cdot \zeta \leq \hat{p} \right\},$$

Theorem 5.2: VaR policy optimization Given any finite MDP with $\alpha \in [0, 1]$, we have

$$\begin{aligned} \max_{\pi \in \Pi} \text{VaR}_\alpha^{\tilde{s} \sim \pi(\tilde{s})} [r(\tilde{s}, \tilde{a}, \tilde{s}')] &= \max_{\pi \in \Pi} \sup_{\zeta \in \Delta_S} \left\{ \min_{s \in \mathcal{S}} \text{VaR}_{\alpha \zeta_s \hat{p}_s^{-1}} [r(s, \tilde{a}, \tilde{s}') \mid \tilde{s} = s] \mid \alpha \cdot \zeta \leq \hat{p} \right\} \\ &= \sup_{\zeta \in \Delta_S} \left\{ \max_{\pi \in \Pi} \min_{s \in \mathcal{S}} \text{VaR}_{\alpha \zeta_s \hat{p}_s^{-1}} [r(s, \tilde{a}, \tilde{s}') \mid \tilde{s} = s] \mid \alpha \cdot \zeta \leq \hat{p} \right\} \\ &= \sup_{\zeta \in \Delta_S} \left\{ \min_{s \in \mathcal{S}} \max_{\pi \in \Pi} \text{VaR}_{\alpha \zeta_s \hat{p}_s^{-1}} [r(s, \tilde{a}, \tilde{s}') \mid \tilde{s} = s] \mid \alpha \cdot \zeta \leq \hat{p} \right\} \end{aligned}$$



Value at Risk (VaR) Value Iteration

$$q_t^*(s, \alpha, a) = \max_{\pi \in \Pi} \text{VaR}_\alpha^\pi \left[\sum_{t=0}^T r(\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}) \right] = \sup_{\zeta \in \Delta_S} \left\{ \min_{s \in \mathcal{S}} \max_{a' \in A} q_{t+1}^*(s', \frac{\alpha \zeta_{s'}}{p_{s a'}}, a') \right\}$$

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