Instability Analysis of Two-Dimensional Bernstein-Greene-Kruskal Modes

J. McClung[1] | M. Franciscovich[2] | K. Germaschewski[1] | C. S. Ng[2]

[1]Department of Physics, University of New Hampshire

[2]Geophysical Institute, University of Alaska Fairbanks

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Abstract

We will present the results of our analysis of the instability of certain two-dimensional (2D) Bernstein-Greene-Kruskal (BGK) modes. BGK modes are steady-state, nonlinear solutions to the Vlasov-Poisson system of equations. In particular, we consider the an ansatz with azimuthal symmetry, a uniform axial magnetic field, and a uniform ion population, where the mode itself manifests as a localized electron hole. Our previous work found that BGK modes following this ansatz were unstable below a critical magnetic field strength, and that azimuthal electrostatic waves formed as part of this instability [1]. We aim to identify and classify the dominant unstable modes of these cases. Unlike our previous work, a particle-in-cell code would likely be ineffective due to the high particle noise, especially in the linear growth phase. We will perform a numerical analysis of the fastest growing unstable modes based on a direct discretization of the linearized Vlasov equation in cylindrical coordinates.

BGK Modes [2]

- steady-state, nonlinear solutions to Vlasov-Poisson system
- can manifest as e.g. electron holes
- normalized Vlasov-Poisson system in cylindrical coordinates

$$\frac{\partial f}{\partial t} + v_\rho \frac{\partial f}{\partial \rho} + \frac{v_\phi}{\rho}\frac{\partial f}{\partial \phi} + \left(\frac{\partial \psi}{\partial \rho} - B_z v_\phi + \frac{v_\phi^2}{\rho}\right)\frac{\partial f}{\partial v_\rho} + \left(\frac{1}{\rho}\frac{\partial \psi}{\partial \phi} + B_z v_\rho - \frac{v_\rho v_\phi}{\rho}\right)\frac{\partial f}{\partial v_\phi} = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho}\frac{\partial \psi}{\partial \phi} = \int f \, \mathrm{d}^2 v - 1$$

Normalization

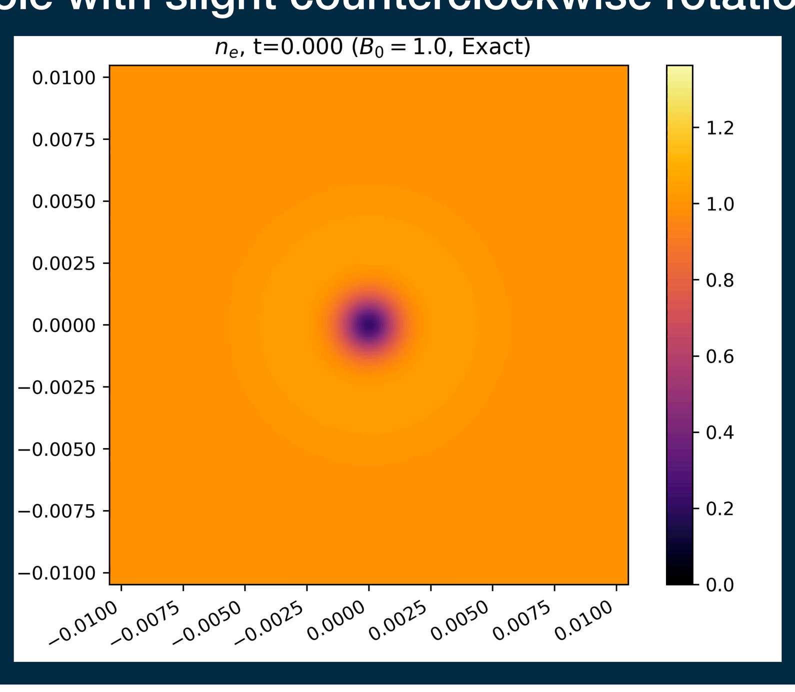
- electron charge = -1
- all 1: m_e, n_i, ε_0, ω_{pe}
- either c or λ_D is 1, depending on the figure

One Set of Solutions [3]

- $h < 1$, $k > 0$, B_z ideally between 0.1 and 10
- solve for ψ numerically:

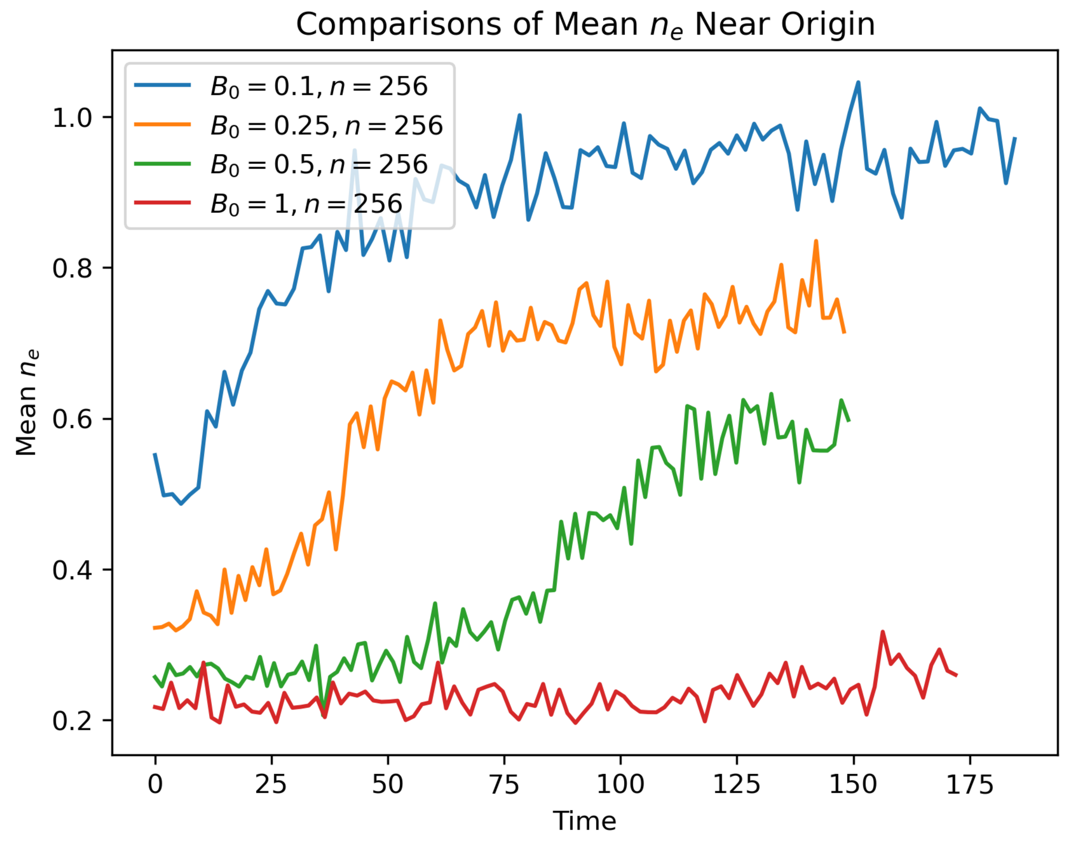
$$f_0(\rho, v_\rho, v_\phi) = \frac{1}{2\pi} e^{-\left(\frac{1}{2}v_\rho^2 + \frac{1}{2}v_\phi^2 - \psi\right)} \left(1 - h e^{-k\left(\rho v_\phi - \frac{1}{2}B_z \rho^2\right)^2}\right)$$

- azimuthally symmetric
- electron hole with slight counterclockwise rotation near hole:



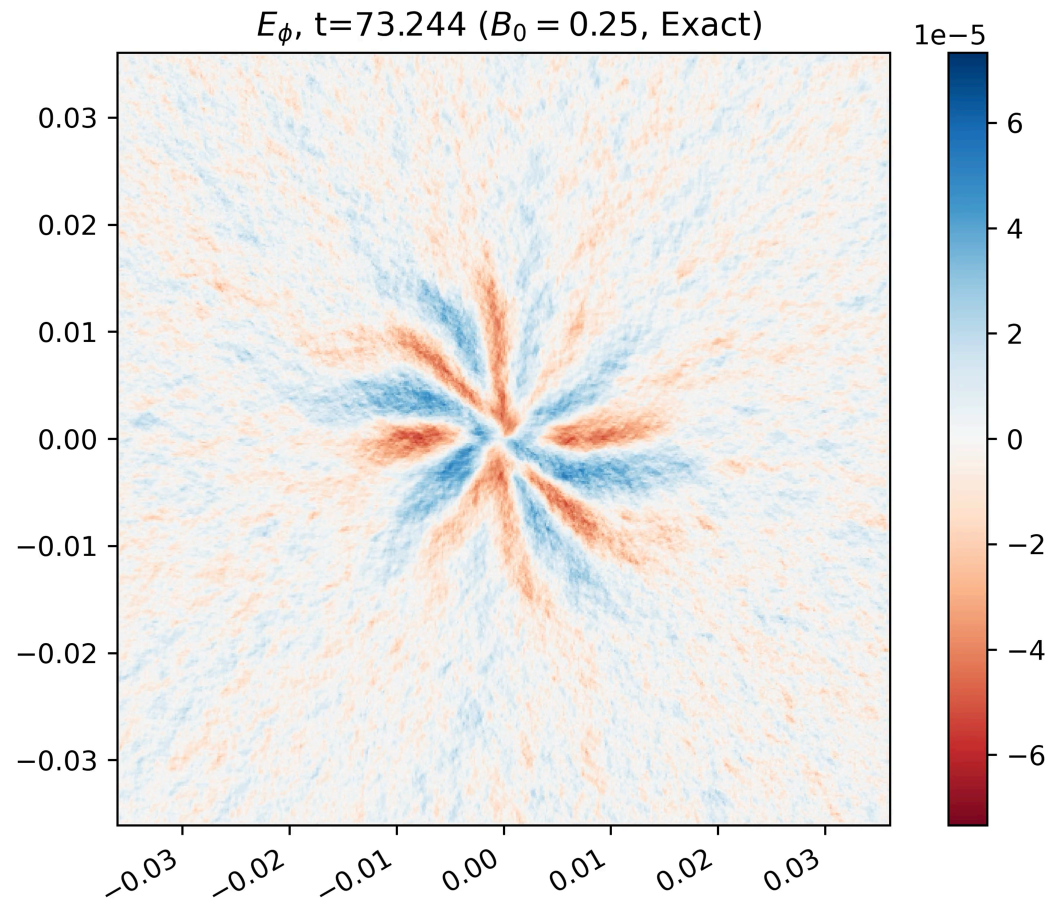
Recent/Motivating Results [1]

- used the Plasma Simulation Code [4]
  - particle-in-cell—too noisy!



Slight Decay

- electron hole fills slightly for $B_0 \le 1$, then stabilizes
- linear phase is too noisy to accurately deduce growth rate

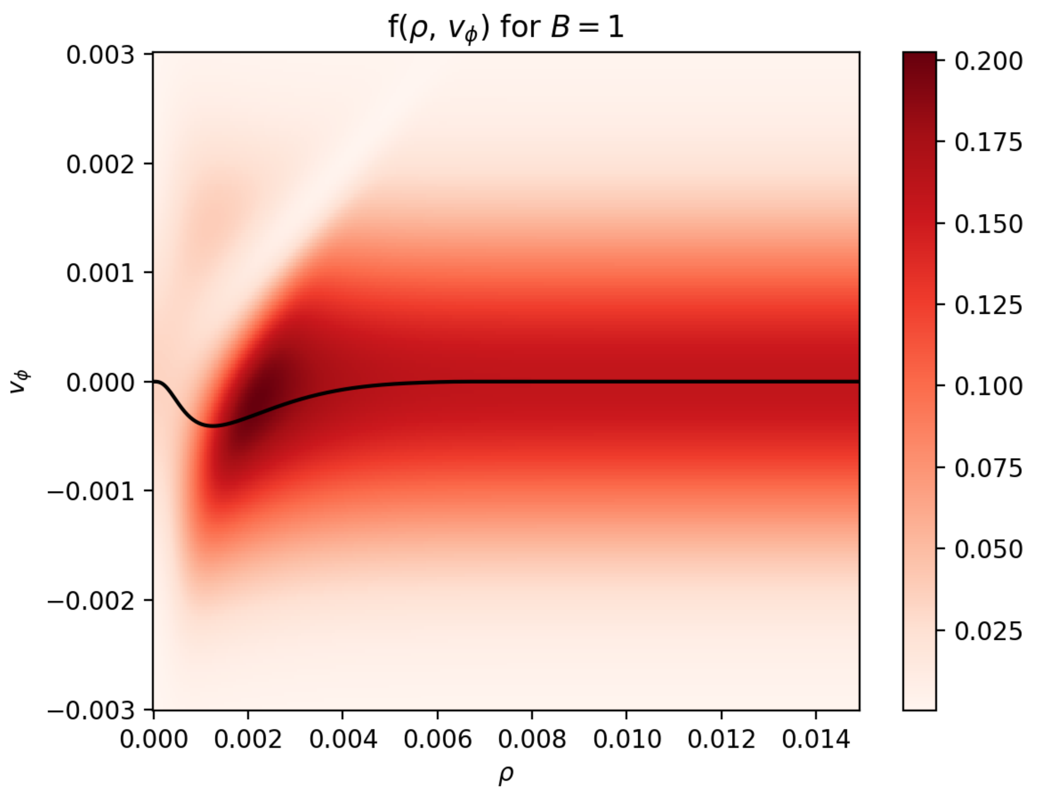


Azimuthal Modes

- azimuthal waves appear during decay
- most apparent in E_ϕ
- bifurcation

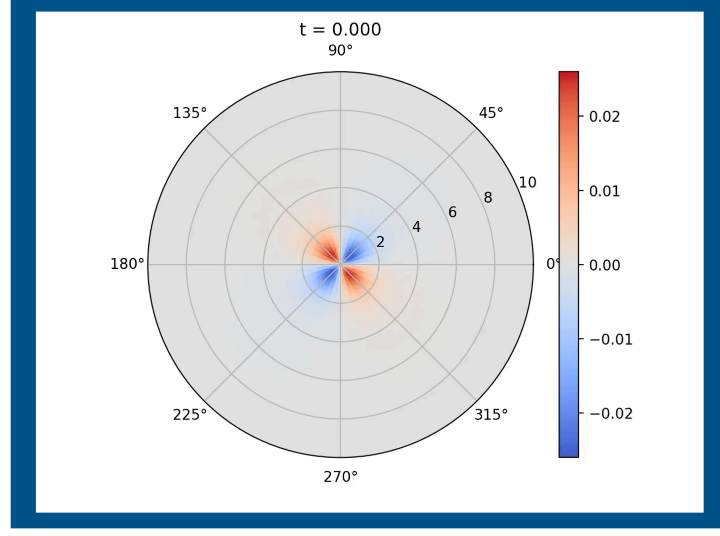
Possible Explanation

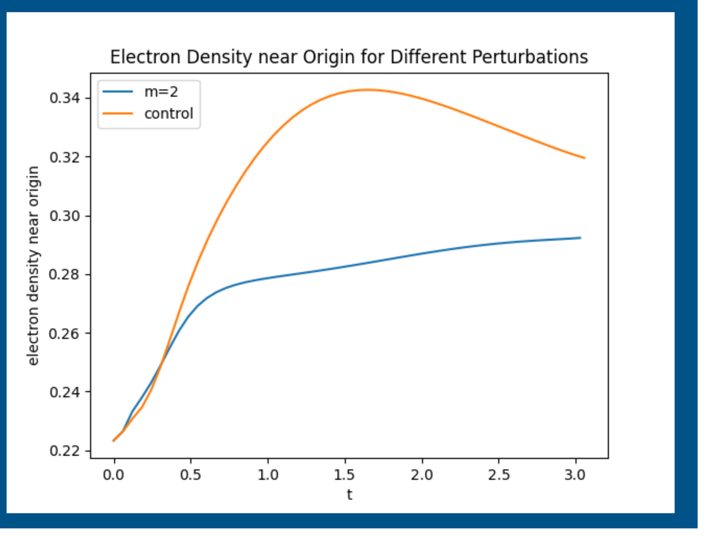
- $f(\rho, v_\phi)$ is bimodal in v_ϕ for small ρ due to "spike"
- classic bump-on-tail instability



Preliminary Results

- $m = 2$ perturbation (left: E_ϕ at t=0)
- problem: unphysical particle injection near origin (right)





New Code

- written in Rust
- cylindrical Vlasov with coordinates ρ, ϕ, v_ρ, v_ϕ
- electrostatic
- static, uniform ion background
- boundary conditions:
  - ρ reflective | ϕ periodic | v_ρ, v_ϕ constant (zero)

Time Evolution: Flux Balance [5,6]

- splitting method: update along ρ and ϕ, then v_ρ and v_ϕ
- general approach for some g satisfying a continuity equation:

$$\frac{\partial}{\partial t} g(t,x) + \frac{\partial}{\partial x}\left(a(t,x) g(t,x)\right) = 0$$

$$Dg[i] := g[i]\frac{|a[i]|\,\Delta t}{\Delta x} + \left(g[i+1] - g[i-1]\right)\frac{a[i]\Delta t}{4\Delta x}\left(1 - \frac{|a[i]|\,\Delta t}{\Delta x}\right)$$

$$g^{\text{next}}[i] = g[i] - Dg[i] + \begin{cases} Dg[i-1], & a[i] > 0 \\ Dg[i+1], & a[i] < 0 \end{cases}$$

Field Solver: FFT and Finite Difference [7]

- Fourier transform charge density q along ϕ
- solve tridiagonal system (below) for each mode m
- inverse transform to obtain ψ
- central difference to obtain E_ρ, E_ϕ

$$A[i] := \frac{1}{\Delta\rho^2}\frac{i+1}{i+0.5}$$
$$B[i] := \frac{1}{\Delta\rho^2}\frac{i}{i+0.5}$$
$$C[i] := \frac{1}{\Delta\rho^2 \Delta\phi^2}\frac{1}{(i+0.5)^2}$$

$$A[i]\hat{\psi}_m[i+1] + B[i]\hat{\psi}_m[i-1] - \left(A[i] + B[i] + 2(1-\cos(m\Delta\phi))C[i]\right)\hat{\psi}_m[i] = -\hat{q}_m[i]$$

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