Position-Space Entropy and Energy Conversion in Collisionless Plasma: A 1D-1V Vlasov Simulation

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Background & Motivation

- In collisionless space plasmas, the energy conversion happens at kinetic scale, where $f_\sigma(\boldsymbol{r},\boldsymbol{v},t)$ is far from local thermodynamic equilibrium (LTE) and first-law thermodynamics breaks down.
- Low-order moments miss the energy that goes into changing the shape of the distribution function.
- **Relative kinetic entropy** is sensitive to the full shape of f_σ → captures this energy conversion (Cassak et al., 2023).

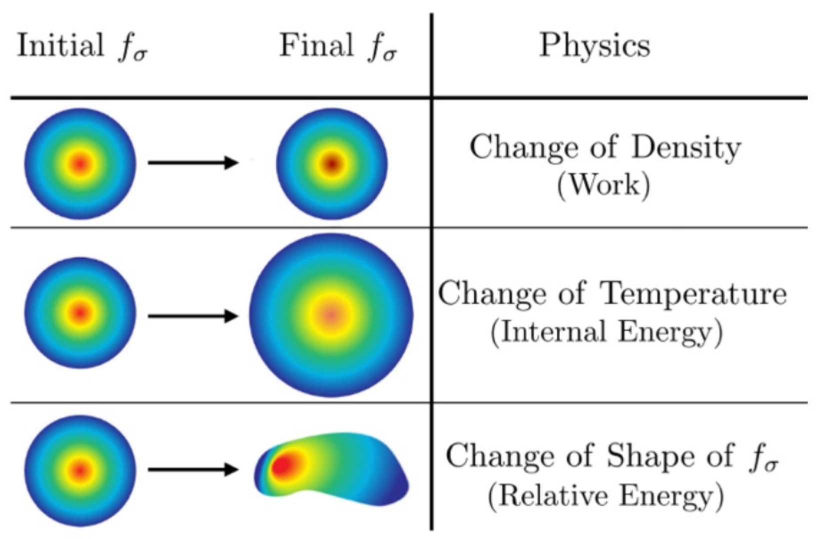

Fig 1: Illustrating of changing shape of the distribution (Cassak et al., 2023).

The theory of kinetic entropy in brief:

- Total Boltzmann kinetic entropy:

$$S_\sigma(t) = -k_B \int\int f_\sigma \ln(f_\sigma) d^3r d^3v\,.$$

- The entropy can be decomposed into position-space and velocity-space entropy (Liang et al., 2019):

$$S_\sigma(t) = S_{\sigma p}(t) + S_{\sigma v}(t).$$

- Entropy density obeys a collisionless continuity equation:

$$\frac{d}{dt}\left(\frac{S_\sigma}{n_\sigma}\right) + \frac{1}{n_\sigma}\nabla\cdot\boldsymbol{J}_\sigma = 0,$$

where $\boldsymbol{J}$ entropy density flux.
- The evolution of relative energy, which quantify energy conversion associated changing shape is

$$\frac{d\mathcal{E}_\sigma}{dt} = \mathcal{T}\frac{d}{dt}\left(\frac{S_{\sigma v}}{n_\sigma}\right).$$

THE PROBLEM:

- Local entropy changes tie to transport and to work done by the fields.
- The entropy density in Cassak et al. (2023) is built by integrating over velocity space.
- However, the electromagnetic force term integrates to zero as divergence theorem in v-space kills it.

THIS WORK:

- Reformulate the kinetic entropy density: integrate over r, not v.
- Boundary term now vanishes over r-space → the force term survives.
- We show preliminary entropy density localized in both r and v, and in both position and velocity space from 1D-1V Vlasov–Poisson simulation.

The Entropy Reformulation

ENTROPY	Localized in position space (integral over $\boldsymbol{v}$)	Localized in velocity space (integral over $\boldsymbol{r}$)
Total position-space	$S_{\sigma p}(t) = k_B\left[N_\sigma \ln\left(\frac{N_\sigma}{\Delta^3 r}\right) - \int n_\sigma \ln(n_\sigma) d^3r\right]$	$S_{\sigma p}(t) = k_B\int\left[\rho_\sigma \ln\left(\frac{\rho_\sigma}{\Delta^3 r}\right) - \int f_\sigma \ln(f_\sigma) d^3r\right]d^3v$
Total velocity-space	$S_{\sigma v}(t) = \int s_{\sigma v} d^3r$	$S_{\sigma v}(t) = -k_B\int \rho_\sigma \ln\left(\frac{\rho_\sigma \Delta^3 v}{N_\sigma}\right)d^3v$
Position-space density	$s_{\sigma p}(\boldsymbol{r},t) = -k_B n_\sigma \ln\left(\frac{n_\sigma \Delta^3 r}{N_\sigma}\right)$	$s_{\sigma p}(\boldsymbol{v},t) = -k_B\int f_\sigma \ln\left(\frac{f_\sigma \Delta^3 r}{\rho_\sigma}\right)d^3r$
Velocity-space density	$s_{\sigma v}(\boldsymbol{r},t) = k_B\left[n_\sigma \ln\left(\frac{n_\sigma}{\Delta^3 v}\right) - \int f_\sigma \ln(f_\sigma) d^3v\right]$	$s_{\sigma v}(\boldsymbol{v},t) = -k_B\int f_\sigma \ln\left(\frac{\rho_\sigma \Delta^3 v}{N_\sigma}\right)d^3r$

Table 1: Entropy formulation: (left column) standard formulation from Liang et al., 2019 and Cassak et al., 2023. (right) Reformulation of entropy following Liang et al., 2019 derivation but integrating over r instead of v.

The standard formulation (as in Cassak et al., 2023) is derived by integrating over velocity space (left column in the table).

In this work, we instead define density localized in velocity:

$$\rho(\boldsymbol{v},t) = \int f(\boldsymbol{r},\boldsymbol{v},t) d^3r\,,$$

and we then derive entropy density by integrating over position space (right column in the table). Taking time derivative to entropy density, the term $\frac{\partial f}{\partial t}$ in the integral can be replaced by Vlasov equation, hence **the electromagnetic force term survives in the entropy evolution** as the integration is done over position space.

Numerical Method

To implement our derivations and see how the force term behaves, we use 1D-1V Vlasov-Poisson simulation (Pezzi et al., 2013). Preliminarily, we initializes the simulation with Maxwellian distribution with a broadband density perturbation:

$$f(x,v,t=0) = f_M(v)[1+\delta n(x)],$$

$$\delta n(x) = \mathcal{E}\sum_{k=1}^{64}\cos(k\delta k_x x + \theta_k),$$

with fixed random phase θ_k normalized so that max $|\delta n| = \mathcal{E} = 10^{-4}$. This excites the full spectrum of Langmuir modes up to $k\Delta k_x = 25.6$; at late times only the slowest-damping fundamental $k_1 = 2\pi/L_x = 0.4$ survives, giving the canonical Landau decay rate $\gamma_L \simeq 0.066$. The physical box $[0, L_x]$ is periodic; the velocity domain $[-v_{max}, +v_{max}]$ uses vanishing-f boundaries, justified by $v_{max} = 6v_{th}$. Entropy densities are computed on the phase-space grid with fixed bin size Δx and Δv.

PARAMETER:

Quantity	L_x	v_{max}	(N_x, N_v)	Δt	t_{final}	$\mathcal{E}$
Value	$5\pi \approx 15.71$	± 6	$(128,1301)$	10^{-2}	1200	10^{-4}

Table 2: 1D1V Vlasov-Poisson simulation parameters.

Results

We first verify the continuity equations in formulation localized in both position and velocity:

$$\frac{\partial s_\sigma(\boldsymbol{r},t)}{\partial t} + \nabla\cdot\boldsymbol{J}_\sigma(\boldsymbol{r},t) = 0,$$

$$\frac{\partial s_\sigma(\boldsymbol{v},t)}{\partial t} + \nabla_v\cdot\boldsymbol{J}_\sigma(\boldsymbol{v},t) = 0.$$

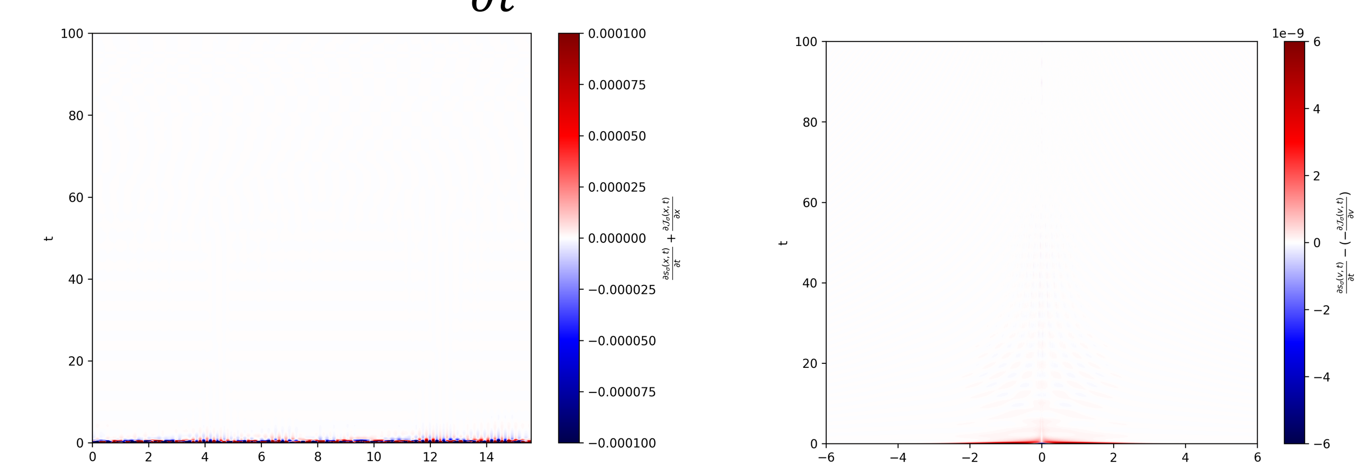

Fig 2: (left) $\frac{\partial s_\sigma(\boldsymbol{r},t)}{\partial t} + \nabla\cdot\boldsymbol{J}_\sigma(\boldsymbol{r},t) = 0$, (right) $\frac{\partial s_\sigma(\boldsymbol{v},t)}{\partial t} + \nabla_v\cdot\boldsymbol{J}_\sigma(\boldsymbol{v},t) = 0$.

Figure 3 shows that the distribution function changes over few time steps and then the wave is then damped away where we cannot see anymore around $t \gtrsim 100$. The right plot also shows quasi-linear plateau formation at the linear-Landau resonant velocity $v_\phi = \omega_R/k \approx 3.21v_{th}$.

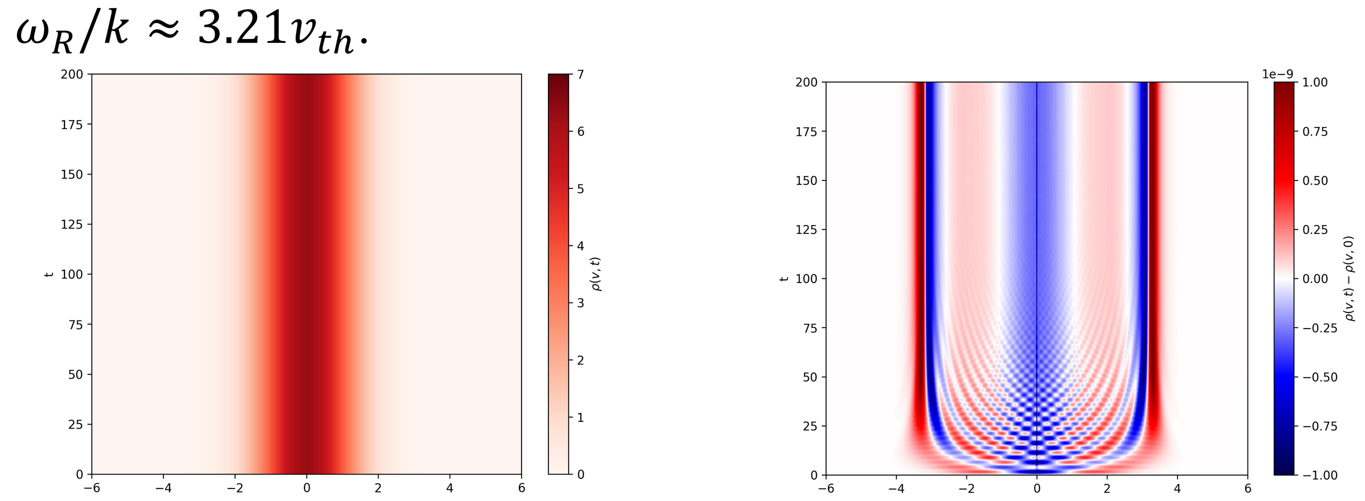

Fig 3: (left) $\rho(v,t)$, (right) $\rho(v,t) - \rho(v,0)$.

Figure 4 shows the time evolution of velocity-space and position-space entropy as a function of velocity. The EM force lies within these terms and contributes to damping during first few time steps. At each time step, both terms are also summed out to zero.

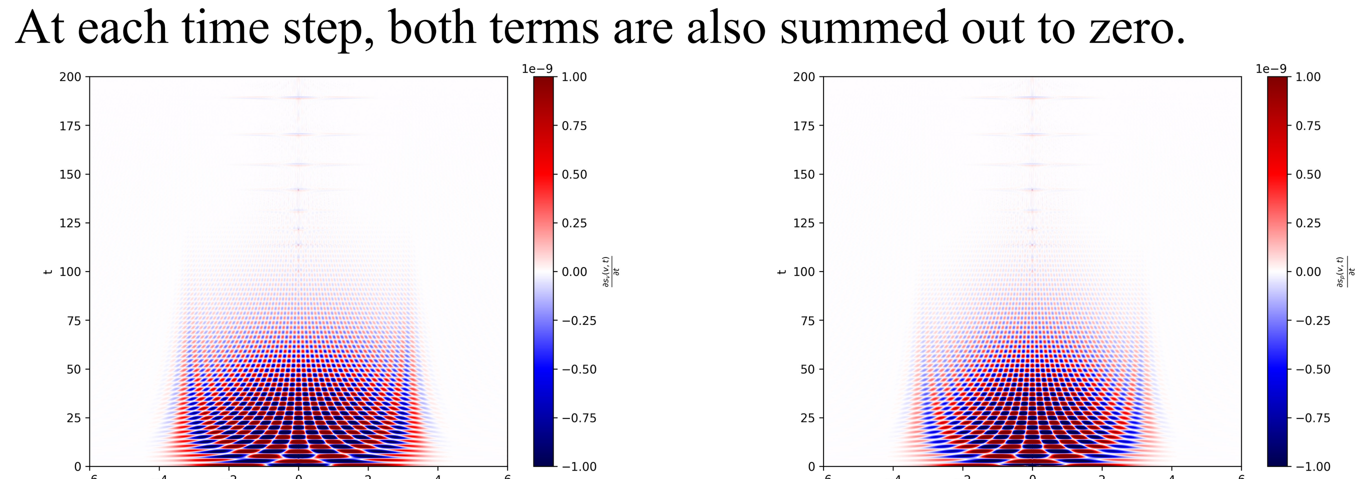

Fig 4: (left) $\frac{ds_{\sigma v}(v,t)}{dt}$, (right) $\frac{ds_{\sigma p}(v,t)}{dt}$.

FUTURE WORKS:

- Plug in Vlasov equation and isolate force terms.
- Extend to magnetized system, included $\boldsymbol{v}\times\boldsymbol{B}$ in Lorentz force.

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